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Grothendieck–Teichmüller and Batalin–Vilkovisky

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Abstract. It is proven that, for any affine supermanifold M equipped with a constant odd symplectic structure, there is a universal action (up to homotopy) of the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$ on the set of quantum BV structures (i.e. solutions of the quantum master equation) on M.

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1. Introduction

Let M be a finite dimensional affine $\mathbb{Z}$-graded manifold M over a field $\mathbb{K}$ equipped with a constant degree 1 symplectic structure ω. In particular, the ring of functions $\mathcal{O}_M$ is a Batalin–Vilkovisky algebra, with Batalin–Vilkovisky operator Δ and bracket $\{\ ,\ \}$. A degree 2 function $S \in \mathcal{O}_M[[u]]$ is a solution of the quantum master equation on M if[1]

$$u\Delta S + \frac{1}{2}\{S, S\} = 0,$$

where u is a formal variable of degree 2. In other words S is a Maurer–Cartan element in the differential graded (dg) Lie algebra $(\mathcal{O}_M[[u]][1], u\Delta, \{\ ,\ \})$.

The Grothendieck–Teichmüller group GRT_1 is a pro-unipotent group introduced by Drinfeld in [3]; we denote its Lie algebra by $\mathfrak{grt}_1$. In this paper we show the following result.

THEOREM 1.1 *There is an L_∞ action of the Lie algebra $\mathfrak{grt}_1$ on the differential graded Lie algebra $(\mathcal{O}_M[[u]][1], u\Delta, \{\ ,\ \})$ by L_∞ derivations. In particular, it follows that there is an action of GRT_1 on the set of gauge equivalence classes of formal solutions of the quantum master equation, i.e. on gauge*

[1]See [11] for an introduction into the geometry of the BV formalism.

equivalence classes of Maurer–Cartan elements in the differential graded Lie algebra $(\hbar\mathcal{O}_M[[u]][[\hbar]][1], u\Delta, \{\ ,\ \})$, *where* $\hbar$ *is a formal deformation parameter of degree 0.*

Our main technical tool is a version of the Kontsevich graph complex, $(\mathsf{GC}_2[[u]], d_u)$ which controls universal deformations of $(\mathcal{O}_M[[u]][1], u\Delta, \{\ ,\ \})$ in the category of L_∞ algebras. Using the main result of [13] we show in Section 2 that

$$H^0(\mathsf{GC}_2[[u]], d_u) \simeq \mathfrak{grt}_1$$

and then use this isomorphism in Section 3 to prove the Main Theorem.

1.1. SOME NOTATION

In this paper $\mathbb{K}$ denotes a field of characteristic 0. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space over $\mathbb{K}$, then $V[k]$ stands for the graded vector space with $V[k]^i := V^{i+k}$. For $v \in V^i$, we set $|v| := i$. The phrase *differential graded* is abbreviated by dg. The n-fold symmetric product of a (dg) vector space V is denoted by $\odot^n V$, and the full symmetric product space by $\odot^\bullet V$. For a finite group G acting on a vector space V, we denote via V^G the space of invariants with respect to the action of G, and by V_G the space of coinvariants $V_G = V/\{gv - v | v \in V, g \in G\}$. As we always work over a field $\mathbb{K}$ of characteristic zero, we have a canonical isomorphism $V_G \cong V^G$.

We use freely the language of operads. For a background on operads we refer to the textbook [10]. For an operad $\mathcal{P}$ we denote by $\mathcal{P}\{k\}$ the unique operad which has the following property: for any graded vector space V there is a one-to-one correspondence between representations of $\mathcal{P}\{k\}$ in V and representations of $\mathcal{P}$ in $V[-k]$; in particular, $\mathcal{E}nd_V\{k\} = \mathcal{E}nd_{V[k]}$.

2. A Variant of the Kontsevich Graph Complex

2.1. FROM OPERADS TO LIE ALGEBRAS

Let $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ be an operad in the category of dg vector spaces with the partial compositions $\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(m+n-1)$, $1 \leq i \leq n$. Then the map

$$\begin{array}{rrcl} [\ ,\ ]: & \mathsf{P}\otimes\mathsf{P} & \longrightarrow & \mathsf{P} \\ & (a\in\mathcal{P}(n), b\in\mathcal{P}(m)) & \longrightarrow & [a,b] := \sum_{i=1}^n a\circ_i b - (-1)^{|a||b|}\sum_{i=1}^m b\circ_i a \end{array}$$

makes the vector space $\mathsf{P} := \prod_{n\geq 1}\mathcal{P}(n)$ into a dg Lie algebra [4,5]. Moreover, the Lie algebra structure descends to the subspace of coinvariants $\mathsf{P}_{\mathbb{S}} := \prod_{n\geq 1}\mathcal{P}(n)_{\mathbb{S}_n}$. Via the identification of invariants and coinvariants $\mathsf{P}_{\mathbb{S}} \cong \mathsf{P}^{\mathbb{S}}$, we furthermore obtain a Lie algebra structure on the space of invariants $\mathsf{P}^{\mathbb{S}} := \prod_{n\geq 1}\mathcal{P}(n)^{\mathbb{S}_n}$ as well.

2.2. AN OPERAD OF GRAPHS AND THE KONTSEVICH GRAPH COMPLEX

For any integers $n \geq 1$ and ≥ 0 we denote by $\mathsf{G}_{n,l}$ a set of graphs,[2] $\{\Gamma\}$, with n vertices and l edges such that (i) the vertices of Γ are labelled by elements of $[n] := \{1, \ldots, n\}$, (ii) the set of edges, $E(\Gamma)$, is totally ordered up to an even permutation. For example, $\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} \in \mathsf{G}_{2,1}$. The group $\mathbb{Z}_2$ acts freely on $\mathsf{G}_{n,l}$ for $l \geq 2$ by changes of the total ordering; its orbit is denoted by $\{\Gamma, \Gamma_{opp}\}$. Let $\mathbb{K}\langle \mathsf{G}_{n,l}\rangle$ be the vector space over a field $\mathbb{K}$ spanned by isomorphism classes, $[\Gamma]$, of elements of $\mathsf{G}_{n,l}$ modulo the relation[3] $\Gamma_{opp} = -\Gamma$, and consider a $\mathbb{Z}$-graded $\mathbb{S}_n$-module,

$$\mathsf{Gra}(n) := \bigoplus_{l=0}^{\infty} \mathbb{K}\langle \mathsf{G}_{n,l}\rangle[l].$$

Note that graphs with two or more edges between any fixed pair of vertices do not contribute to $\mathsf{Gra}(n)$, so that we could have assumed right from the beginning that the sets $\mathsf{G}_{n,l}$ do not contain graphs with multiple edges. The $\mathbb{S}$-module, $\mathsf{Gra} := \{\mathsf{Gra}(n)\}_{n\geq 1}$, is naturally an operad with the operadic compositions given by

$$\begin{aligned} \circ_i : \mathsf{Gra}(n) \otimes \mathsf{Gra}(m) &\longrightarrow \mathsf{Gra}(m+n-1) \\ \Gamma_1 \otimes \Gamma_2 &\longrightarrow \textstyle\sum_{\Gamma \in \mathsf{G}^i_{\Gamma_1,\Gamma_2}} (-1)^{\sigma_\Gamma} \Gamma \end{aligned}$$

where $\mathsf{G}^i_{\Gamma_1,\Gamma_2}$ is the subset of $\mathsf{G}_{n+m-1,\#E(\Gamma_1)+\#E(\Gamma_2)}$ consisting of graphs, Γ, satisfying the condition: the full subgraph of Γ spanned by the vertices labeled by the set $\{i, i+1, \ldots, i+m-1\}$ is isomorphic to Γ_2, and the quotient graph, Γ/Γ_2, obtained by contracting that subgraph to a single vertex, is isomorphic to Γ_1. The sign $(-1)^{\sigma_\Gamma}$ is determined by the equality

$$\bigwedge_{e\in E(\Gamma)} e = (-1)^{\sigma_\Gamma} \bigwedge_{e'\in E(\Gamma_1)} e' \wedge \bigwedge_{e''\in E(\Gamma_2)} e''.$$

The unique element in $\mathsf{G}_{1,0}$ serves as the unit element in the operad Gra. The associated Lie algebra of $\mathbb{S}$-invariants, $((\mathsf{Gra}\{-2\})^{\mathbb{S}}, [\ ,\ ])$ is denoted, following notations of [13], by fGC_2. Its elements can be understood as graphs from $\mathsf{G}_{n,l}$ but with labeling of vertices forgotten, e.g.

$$\bullet\!\!-\!\!\!-\!\!\bullet = \frac{1}{2}\left(\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} + \overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\right) \in \mathsf{fGC}_2.$$

The cohomological degree of a graph with n vertices and l edges is $2(n-1)-l$. It is easy to check that $\bullet\!\!-\!\!\!-\!\!\bullet$ is a Maurer–Cartan element in the Lie algebra fGC_2. Hence, we obtain a dg Lie algebra

$$(\mathsf{fGC}_2, [\ ,\ ], d := [\bullet\!\!-\!\!\!-\!\!\bullet,\ ]).$$

[2] A *graph* Γ is, by definition, a 1-dimensional CW-complex whose 0-cells are called *vertices* and 1-dimensional cells are called *edges*. The set of vertices of Γ is denoted by $V(\Gamma)$ and the set of edges by $E(\Gamma)$.

[3] Abusing notations we identify from now an equivalence class $[\Gamma]$ with any of its representative Γ.

One may define a dg Lie subalgebra, GC_2, spanned by connected graphs with at least trivalent vertices and no edges beginning and ending at the same vertex. It is called the *Kontsevich graph complex* [7]. We leave it to the reader to verify that the subspace GC_2 is indeed closed under both the differential and the Lie bracket. We refer to [13] for a detailed explanation of why studying the dg Lie subalgebra GC_2 rather than full Lie algebra fGC_2 should be enough for most purposes. The cohomologies of GC_2 and fGC_2 were partially computed in [13].

THEOREM 2.1 ([13]). *(i)* $H^0(\mathsf{GC}_2, d) \simeq \mathfrak{grt}_1$. *(ii)* For any negative integer i, $H^i(\mathsf{GC}_2, d) = 0$.

We shall introduce next a new graph complex which is responsible for the action of GRT_1 on the set of quantum master functions on an odd symplectic supermanifold.

2.3. A VARIANT OF THE KONTSEVICH GRAPH COMPLEX

The graph  $\in \mathsf{fGC}_2$ has degree -1 and satisfies

[ , ] = [ ,] = 0.

Let u be a formal variable of degree 2 and consider the graph complex $\mathsf{fGC}_2[[u]]$ with the differential

$d_u := d + u\Delta$, where $\Delta :=$ [ ,] .

The subspace $\mathsf{GC}_2[[u]] \subset \mathsf{fGC}_2[[u]]$ is a subcomplex of $(\mathsf{fGC}_2[[u]], d_u)$.

PROPOSITION 2.1. $H^0(\mathsf{GC}_2[[u]], d_u) \simeq \mathfrak{grt}_1$ *and* $H^{\leq -1}(\mathsf{GC}_2[[u]]) = 0$.

Proof. Consider a decreasing filtration of $\mathsf{GC}_2[[u]]$ by the powers in u. The first term of the associated spectral sequence is

$$\mathcal{E}_1 = \bigoplus_{i\in\mathbb{Z}} \mathcal{E}_1^i, \qquad \mathcal{E}_1^i = \prod_{p\geq 0} H^{i-2p}(\mathsf{GC}_2, d)u^p$$

with the differential equal to $u\Delta$. As $H^0(\mathsf{GC}_2, d) \simeq \mathfrak{grt}_1$ and $H^{\leq -1}(\mathsf{GC}_2, d) = 0$, one gets the desired result. $H^0(\mathsf{fGC}_2[[u]], d_u) \simeq \mathfrak{grt}_1$.

The projections $(\mathsf{GC}_2[[u]], d_u) \to (\mathsf{GC}_2, d)$ and $(\mathsf{fGC}_2[[u]], d_u) \to (\mathsf{fGC}_2, d)$ sending u to 0 are maps of Lie algebras and induce isomorphisms in degree 0 cohomology. Since the isomorphisms of Theorem 2.1 (i) are maps of Lie algebras as shown in [13], so are the maps in the above Proposition. $\square$

Remark 2.2. Let σ be an element of $\mathfrak{grt}_1$ and let $\Gamma_\sigma^{(0)}$ be any cycle representing the cohomology class σ in the graph complex (GC_2, d). Then one can construct a cocycle,

$$\Gamma_\sigma^u = \Gamma_\sigma^{(0)} + \Gamma_\sigma^{(1)} u + \Gamma_\sigma^{(2)} u^2 + \Gamma_\sigma^{(3)} u^3 + \dots, \tag{1}$$

representing the cohomology class $\sigma \in \mathfrak{grt}_1$ in the complex $(\mathsf{GC}_2[[u]], d_u)$ by the following induction:

1st step: As $d\Gamma_\sigma^{(0)} = 0$, we have $d(\Delta\Gamma_\sigma^{(0)}) = 0$. As $H^{-1}(\mathsf{GC}_2, d) = 0$, there exists $\Gamma_\sigma^{(1)}$ of degree -2 such that $\Delta\Gamma_\sigma^{(0)} = -d\Gamma_\sigma^{(1)}$ and hence

$$(d + u\Delta)(\Gamma_\sigma^{(0)} + \Gamma_\sigma^{(1)} u) = 0 \bmod O(u^2).$$

n-th step: Assume we have constructed a polynomial $\sum_{i=1}^n \Gamma_\sigma^{(i)} u^i$ such that

$$(d + u\Delta) \sum_{i=1}^n \Gamma_\sigma^{(i)} u^i = 0 \bmod O(u^{n+1}).$$

Then $d(\Delta\Gamma_\sigma^{(n)}) = 0$, and, as $H^{-2n-1}(\mathsf{GC}_2, d) = 0$, there exists a graph $\Gamma_\sigma^{(n+1)}$ in GC_2 of degree $-2n-2$ such that $\Delta\Gamma_\sigma^{(n)}) = -d\Gamma_\sigma^{(n+1)}$. Hence, $(d + u\Delta) \sum_{i=1}^{n+1} \Gamma_\sigma^{(i)} u^i = 0 \bmod O(u^{n+2})$.

3. Quantum BV Structures on Odd Symplectic Manifolds

3.1. MAURER–CARTAN ELEMENTS AND GAUGE TRANSFORMATIONS

Let $(\mathfrak{g} = \oplus_{i\in\mathbb{Z}} \mathfrak{g}^i, [\ ,\ ], d)$ be a dg Lie algebra and consider the dg Lie algebra $\mathfrak{g}_\hbar := \hbar\mathfrak{g}[[\hbar]] =: \oplus_{i\in\mathbb{Z}} \mathfrak{g}_\hbar^i$, where $\hbar$ is a formal deformation parameter. The group $G := \exp(\mathfrak{g}_\hbar^0)$ (which is, as a set, $\mathfrak{g}_\hbar^0$ equipped with the standard Baker–Campbell–Hausdorff multiplication) acts on $\mathfrak{g}_\hbar^1$,

$$\gamma \to \exp(h)\cdot\gamma := \mathrm{e}^{\mathrm{ad}_h}\gamma - \frac{\mathrm{e}^{\mathrm{ad}_h} - 1}{\mathrm{ad}_h}\mathrm{d}h,$$

preserving its subset of Maurer–Cartan elements

$$\mathcal{MC}(\mathfrak{g}_\hbar) = \left\{\gamma \in \mathfrak{g}_\hbar^1 | d\gamma + \frac{1}{2}[\gamma, \gamma] = 0\right\}.$$

We call the G-orbits in $\mathcal{MC}(\mathfrak{g}_\hbar)$ the gauge equivalence classes of Maurer–Cartan elements.

The group of L_∞ automorphism of $\mathfrak{g}$ acts on $\mathcal{MC}(\mathfrak{g}_\hbar)$ by the formula

$$F\cdot\gamma := \sum_{n\geq 1} \frac{1}{n!} F_n(\gamma, \dots, \gamma)$$

where F_n is the n-th component of the L_∞ morphism. In particular, let f be an L_∞ derivation of $\mathfrak{g}$ without linear term. It exponentiates to an L_∞ automorphism $\exp(f)$ of $\mathfrak{g}$, which acts on $\mathcal{MC}(\mathfrak{g}_\hbar)$, and in particular on the set of gauge equivalence classes. By a small calculation one may check that if we change f by homotopy, i.e. by adding $\mathrm{d}h$ for some degree 0 element h of the Chevalley–Eilenberg complex of $\mathfrak{g}$, then the induced actions of $\exp(f)$ and $\exp(f+\mathrm{d}h)$ on the set of gauge equivalence classes agree.

3.2. QUANTUM BV MANIFOLDS

Let M be a $\mathbb{Z}$-graded manifold equipped with an odd symplectic structure ω (of degree 1). There always exist so-called Darboux coordinates, $(x^a, \psi_a)_{1\le a\le n}$, on M such that $|\psi_a| = -|x^a|+1$ and $\omega = \sum_a \mathrm{d}x^a \wedge \mathrm{d}\psi_a$. The odd symplectic structure makes, in the obvious way, the structure sheaf into a Lie algebra with brackets, $\{\ ,\ \}$, of degree -1. A less obvious fact is that ω induces a degree -1 differential operator, Δ_ω, on the invertible sheaf of semidensities, $\mathrm{Ber}(M)^{\frac{1}{2}}$ [6]. Any choice of a Darboux coordinate system on M defines an associated trivialization of the sheaf $\mathrm{Ber}(M)^{\frac{1}{2}}$; if one denotes the associated basis section of $\mathrm{Ber}(M)^{\frac{1}{2}}$ by $D_{x,\psi}$, then any semidensity D is of the form $f(x,\psi)D_{x,\psi}$ for some smooth function $f(x,\psi)$, and the operator Δ_ω is given by

$$\Delta_\omega(f(x,\psi)D_{x,\psi}) = \sum_{a=1}^{n} \frac{\partial^2 f}{\partial x^a \partial \psi_a} D_{x,\psi}.$$

Let u be a formal parameter of degree 2. A *quantum master function* on M is a u-dependent semidensity D which satisfies the equation

$$\Delta_\omega D = 0$$

and which admits, in some Darboux coordinate system, a form

$$D = \mathrm{e}^{\frac{S}{u}} D_{x,\psi},$$

for some $S \in \mathcal{O}_M[[u]]$ of total degree 2, where $\mathcal{O}_M$ is the algebra of functions on M. In the literature it is this formal power series in u which is often called a quantum master function. Let us denote the set of all quantum master functions on M by $\mathcal{QM}(M)$. It is easy to check that the equation $\Delta_\omega D = 0$ is equivalent to the following one,

$$u\Delta S + \frac{1}{2}\{S, S\} = 0, \tag{2}$$

where $\Delta := \sum_{a=1}^{n} \frac{\partial^2}{\partial x^a \partial \psi_a}$. This equation is often called the *quantum master equation*, while a triple $(M, \omega, S \in \mathcal{QM}(M))$ a *quantum BV manifold*.

Let us assume from now on that M is affine or formal (i.e., we work with ∞-jets of functions at some point) and that a particular Darboux coordinate system is fixed on M up to affine transformations[4] so that the algebra of function on M is $\mathcal{O}_M \cong \mathbb{K}[x^a, \psi_a]$ or $\mathcal{O}_M \cong \mathbb{K}[[x^a, \psi_a]]$.

For later reference we will also consider solutions of (2) that depend on a formal deformation parameter $\hbar$ of degree 0, $S \in \hbar\mathcal{O}_M[[u]][[\hbar]]$. We will call the set of such S the *set of formal solutions of the quantum master equation* and denote it by $\mathcal{QM}_\hbar(M)$.

3.3. AN ACTION OF GRT_1 ON QUANTUM MASTER FUNCTIONS

The constant odd symplectic structure on M makes $\mathcal{O}_M$ into a representation

$$\begin{aligned} \rho: \mathsf{Gra}(n) &\longrightarrow \mathsf{End}_V(n) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}_M^{\otimes n}, \mathcal{O}_M) \\ \Gamma &\longrightarrow \Phi_\Gamma \end{aligned} \tag{3}$$

of the operad Gra as follows:

$$\begin{aligned} &\Phi_\Gamma(S_1, \ldots, S_n) \\ &\quad := \pi\Big(\prod_{e \in E(\Gamma)} \Delta_e(S_1(x_{(1)}, \psi_{(1)}, u) \otimes S_2(x_{(2)}, \psi_{(2)}, u) \otimes \ldots \otimes S_n(x_{(n)}, \psi_{(n)}, u))\Big) \end{aligned}$$

where, for an edge e connecting vertices labeled by integers i and j,

$$\Delta_e = \sum_{a=1}^{n} \frac{\partial}{\partial x^a_{(i)}} \frac{\partial}{\partial \psi_{a(j)}} + \frac{\partial}{\partial \psi_{a(i)}} \frac{\partial}{\partial x^a_{(j)}}$$

with the subscript (i) or (j) indicating that the derivative operator is to be applied to the i-th of j-th factor in the tensor product. The symbol π in (4) denotes the multiplication map,

$$\begin{aligned} \pi: \qquad V^{\otimes n} &\longrightarrow V \\ S_1 \otimes S_2 \otimes \ldots \otimes S_n &\longrightarrow S_1 S_2 \cdots S_n. \end{aligned}$$

Let $V := \mathcal{O}_M[[u]]$. Then by u-linear extension, we obtain a continuous representation (in the category of topological $\mathbb{K}[[u]]$-modules)

$$\mathsf{Gra}[[u]] \longrightarrow \mathsf{End}_V = \mathrm{Hom}_{\mathrm{cont}}(V^{\otimes \cdot}, V). \tag{4}$$

The space $V[1]$ is a topological dg Lie algebra with differential $u\Delta$ and Lie bracket $\{\ ,\ \}$. These data define a Maurer–Cartan element, $\gamma_{\mathcal{QM}} := u\Delta \oplus \{\ ,\ \}$ in the

[4]This is not a serious loss of generality as any quantum master equation can be represented in the form (2). Our action of GRT_1 on $\mathcal{QM}_\hbar(M)$ depends on the choice of an affine structure on M in exactly the same way as the classical Kontsevich's formula for a universal formality map [8] depends on such a choice. A choice of an appropriate affine connection on M and methods of the paper [2] can make our formulae for the GRT_1 action invariant under the group of symplectomorphisms of (M, ω); we do not address this *globalization* issue in the present note.

Lie algebra $(\mathsf{End}_V\{-2\})^{\mathbb{S}} \subset CE^\bullet(V,V)$, where $CE^\bullet(V,V)$ is the Lie algebra of coderivations

$$CE^\bullet(V,V) = (\mathrm{Coder}(\odot^{\bullet\geq 1}(V[2])), [\ ,\ ]) \quad \text{with}$$
$$CE^\bullet(V,V)_{(m)} := \mathrm{Hom}(\odot^{\bullet\geq m+1}(V[2]), V[2]),$$

of the standard graded co-commutative coalgebra, $\odot^{\bullet\geq 1}(V[2])$, co-generated by a vector space V. The set $\mathcal{MC}(CE^\bullet(V,V))$ can be identified with the set of L_∞ structures on the space $V[1]$.

The map sending an operad $\mathcal{P}$ to the Lie algebra of invariants $\prod_n \mathcal{P}\{-2\}(n)^{\mathbb{S}_n}$ is functorial. Hence, from the representation (4) we obtain a map of graded Lie algebras

$$\mathsf{fGC}_2[[u]] \cong (\mathsf{Gra}\{-2\}[[u]])^{\mathbb{S}} \to (\mathsf{End}_V\{-2\})^{\mathbb{S}} \subset CE^\bullet(V,V)$$

One checks that the Maurer–Cartan element

$\bullet\!\!-\!\!\!-\!\!\bullet + u$  $\in \mathsf{fGC}_2[[u]]$

is sent to the Maurer–Cartan element $\gamma_{\mathcal{QM}} \in CE^\bullet(V,V)$. Hence, we obtain a morphism of dg Lie algebras

$$(\mathsf{fGC}_2[[u]], [\ ,\ ], d_h) \longrightarrow (CE^\bullet(V,V), [\ ,\ ], \delta := [\gamma_{\mathcal{QM}}, \ ]),$$

and by restriction a morphism

$$\Phi\colon (\mathsf{GC}_2[[u]], [\ ,\ ], d_h) \longrightarrow (CE^\bullet(V,V), [\ ,\ ], \delta := [\gamma_{\mathcal{QM}}, \ ]),$$

Hence, we also obtain a morphism of their cohomology groups,

$$\mathfrak{grt}_1 \simeq H^0(\mathsf{GC}_2[[u]], d_u) \longrightarrow H^0(CE^\bullet(V,V), \delta).$$

Let σ be an arbitrary element in $\mathfrak{grt}_1$ and let Γ^u_σ be a cocycle representing σ in the graph complex $(\mathsf{GC}_2[[u]], d_u)$. We may assume that Γ^u_σ consists of graphs with at least 4 vertices; see [13]. Then the element $\Phi(\Gamma^u_\sigma)$ describes an L_∞ derivation of the Lie algebra $V[1]$ without the linear term. By exponentiation we obtain an L_∞ automorphism,

$$F^\sigma = \{F^\sigma_n : \odot^n V \longrightarrow V[2-2n]\}_{n\geq 1},$$

of the dg Lie algebra $(V[1], u\Delta, \{\ ,\ \})$ with $F^\sigma_1 = \mathrm{Id}$. Hence, for any formal quantum master function $S \in \mathcal{QM}_\hbar(M)$ the series

$$S^\sigma := S + \sum_{n\geq 2} \frac{1}{n!} F^\sigma_n(S, \ldots, S)$$

gives again a formal quantum master function.[5] The induced action on gauge equivalence classes of such functions is well defined, i.e. it does not depend on the representative Γ^u_σ chosen. This is the acclaimed homotopy action of GRT_1 on $\mathcal{QM}_\hbar(M)$ for any affine odd symplectic manifold M.

Remark 3.1. As pointed out by one of the referees, there is also a stronger notion of "homotopy action" that holds in our setting. We will only consider the infinitesimal version. Then, we do not only have a Lie algebra morphism $\mathfrak{grt}_1 \to H^0(CE^\bullet(V,V))$, but an L_∞ morphism $\mathfrak{grt}_1 \to CE^\bullet(V,V)$ as follows. First, consider the truncated version $(\mathsf{GC}_2[[u]])^{tr}$ of the dg Lie algebra $\mathsf{GC}_2[[u]]$, which is by definition the same as $\mathsf{GC}_2[[u]]$ in negative degrees, zero in positive degrees, and consists of the degree zero cocycles in degree zero. By Proposition 2.1 the canonical projection $(\mathsf{GC}_2[[u]])^{tr} \to \mathfrak{grt}_1$ is a quasi-isomorphism. Hence we can obtain the desired L_∞ morphism $\mathfrak{grt}_1 \to CE^\bullet(V,V)$ by lifting the zig-zag

$$\mathfrak{grt}_1 \xleftarrow{\sim} (\mathsf{GC}_2[[u]])^{tr} \longrightarrow CE^\bullet(V,V).$$

This proves the first claim of the main Theorem.

Remark 3.2. It is a well-known result due to Tamarkin [12] that the Grothendieck Teichmüller group GRT_1 acts on the operad of chains of the little disks operad. In fact, one can show that this GRT_1 action extends to an action on the operad of chains of the framed little disks operad, which is quasi-isomorphic to the Batalin–Vilkovisky operad. Hence, one obtains in particular an action of GRT_1 on the set of Batalin–Vilkovisky algebra structures on any vector space, and on their deformations, up to homotopy. In our setting the algebra $\mathcal{O}_M$ is an algebra over the framed little disks operad. Any solution $S = S_0 + uS_1 + u^2S_2 + \cdots$ of the master equation (2) yields a deformation of the Batalin–Vilkovisky structure on $\mathcal{O}_M$, up to homotopy. Concretely, to S one may associate a BV_∞^{com}-structure (see [9] or [1, section 5.3]), whose n-th order "BV" operator is defined as $\Delta_n := [S_n, \cdot]$ (notation as in [1, section 5.3]). The GRT_1 action on solutions of the master equation described above can hence be seen as a shadow of this more general action of GRT_1 on the framed little disks operad. However, we leave the details to elsewhere.

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[5]The series trivially converges since we work in the formal setting, i.e. $S = \hbar(\cdots)$. Ideally, of course, one hopes to have a nonzero convergence radius in $\hbar$, but we cannot guarantee this.

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